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Six-Vertex Model with Rotated Boundary Conditions

Masafumi Fujimoto[1]

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We investigate the six-vertex model on a square lattice rotated through an arbitrary angle with respect to the coordinate axes, a model recently introduced by Litvin and Priezzhev. Auxiliary vertices are used to define an inhomogeneous system which leads to a one-parameter family of commuting transfer matrices. A product of commuting transfer matrices can be interpreted as a transfer matrix acting on zigzag walls in the rotated system. Using an equation for commuting transfer matrices, we calculate their eigenvalues. Finite-size properties of the model are discussed from the viewpoint of the conformal field theory.

KEY WORDS: Six-vertex model; transfer matrix; lattice rotation; auxiliary vertex; conformal invariance; XXZ-Heisenberg chain.

1. INTRODUCTION

Litvin and Priezzhev[1] (LP) investigated the six-vertex model defined on a square lattice rotated through an arbitrary angle φ with respect to the coordinate axes (Fig. 1). Using a random-walk formalism, LP derived the Bethe ansatz equation[2–4] for general φ. In the case of the ice model the Bethe ansatz equation was solved numerically to show that several cases of φ give the same value of the per-site entropy $s_l = (3/2)\ln(4/3)$[3].

LP indicated that, when we consider finite-size properties of the model, lattice rotations offer more interesting problems: For example, if we assume a conformally invariant model[5–7] wrapped on a torus of size $l \times l'(l' \gg l \gg 1)$, it follows that the free energy F must be of the form[8]

$$F = ll'f - (l'/l)(\pi c/6) + \cdots \tag{1.1}$$

[1] Department of Physics, Nara Medical University, Kashihara, Nara 634, Japan.

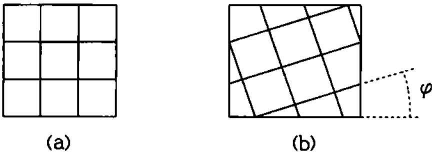

Fig. 1. A square lattice (a) in the natural orientation and (b) rotated through φ with respect to the coordinate axes.

where f is the per-site free energy and c is a universal number called the central charge. For the six-vertex model the value of c has been determined in analyses for $\varphi = 0$[9-11] and $\pi/4$.[12] It is desirable to show that the l'/l correction term in (1.1) is invariant under lattice rotations.

Recently, we developed a new method for solving interacting hard squares on a rotated lattice.[13] We introduced auxiliary faces into a rotated system to relate it to an inhomogeneous one in the natural orientation $\varphi = 0$ (Fig. 1a). The inhomogeneous system was investigated by a commuting transfer matrices argument.[4]

The auxiliary faces (or vertices) method is a general one, applicable to a wide class of solvable models. In this paper we use it to analyze the six-vertex model on a rotated lattice. We assume a gapless regime of the model. Finite-size properties are discussed from the viewpoint of the conformal field theory.

The present paper is organized as follows. In Section 2 we introduce the six-vertex model. In Section 3 we define an inhomogeneous system by the use of auxiliary vertices. The inhomogeneous system leads to a one-parameter family of commuting transfer matrices. We investigate an equation for commuting transfer matrices to determine their eigenvalues. Using calculated eigenvalues, we analyze the six-vertex model on a rotated lattice. Section 4 is devoted to a summary and discussion.

2. SIX-VERTEX MODEL: GAPLESS REGIME

In the six-vertex model an arrow is placed on every edge of a square lattice so that two arrows point into and out of each site (or vertex).[4] The Boltzmann weight is assigned on each vertex depending on the arrow configurations around it. For definiteness we assume a square lattice in the natural orientation here. Then, arrow configurations are represented by associating an arrow spin α_j with each edge j; $\alpha_j = +1$ if the corresponding arrow points up or to the right, and $\alpha_j = -1$ otherwise.

There are six possible arrow configurations around a vertex (Fig. 2a). By $W(\nu, \alpha \mid \beta, \mu)$ we denote the Boltzmann weight of a vertex with

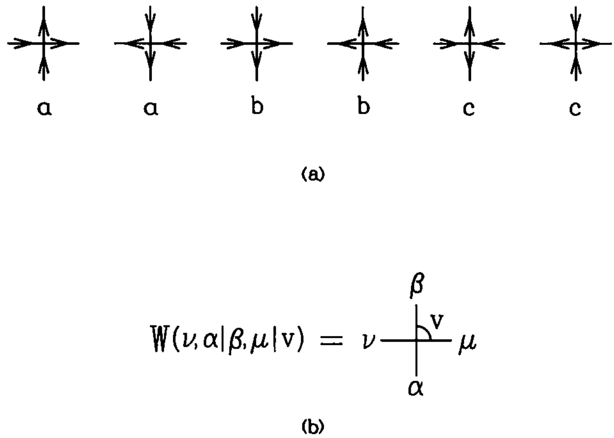

Fig. 2. Arrow configurations around a vertex.

arrow-spins ν, α, μ, and β counterclockwise starting from the west edge (Fig. 2b). The Boltzmann weights W are parametrized as

$$\begin{aligned} a &= W(++\,|\,++) = W(--\,|\,--) = \sinh(\lambda/2 - v)/\sinh\lambda \\ b &= W(+-\,|\,-+) = W(-+\,|\,+-) = \sinh(\lambda/2 + v)/\sinh\lambda \\ c &= W(+-\,|\,+-) = W(-+\,|\,-+) = 1 \end{aligned} \tag{2.1}$$

In this paper analyses are restricted to a gapless regime, where the crossing parameter λ and the spectral parameter v are imaginary numbers:

$$\begin{aligned} \lambda &= -i\bar{\lambda}, \qquad 0 < \bar{\lambda} < \pi \\ v &= -i\bar{v}, \qquad -\bar{\lambda}/2 < \bar{v} < \bar{\lambda}/2 \end{aligned} \tag{2.2}$$

When $\bar{\lambda} = 2\pi/3$ and $\bar{v} = 0$, the model reduces to the ice model.[3,4] The Boltzmann weights W satisfy the Yang–Baxter relation[4,14,15]

$$\begin{aligned} &\sum_{\gamma,\mu'',\nu''} W(\mu,\alpha|\,\gamma,\mu''\,|v)\, W(\nu,\gamma|\,\beta,\nu''\,|v')\, W(\nu'',\mu''|\,\nu',\mu'\,|v'') \\ &\quad = \sum_{\gamma,\mu'',\nu''} W(\nu,\mu|\,\nu'',\mu''\,|v'')\, W(\mu'',\alpha|\,\gamma,\mu'\,|v')\, W(\nu'',\gamma|\,\beta,\nu'\,|v) \end{aligned} \tag{2.3}$$

for all $\alpha, \beta, \mu, \nu, \mu', \nu' = \pm 1$ with $v' = v + v'' + \lambda/2$. The following properties are also satisfied by W:[14,15] the standard initial condition

$$W(\nu,\alpha|\,\beta,\mu\,|-\lambda/2) = \delta(\nu,\beta)\,\delta(\alpha,\mu) \tag{2.4}$$

822/82/5-6-20

and crossing symmetry

$$W(\nu, \alpha|\ \beta, \mu\ |-v) = W(\alpha,\ -\mu|\ -\nu, \beta\ |v) \tag{2.5}$$

Using (2.5) in (2.4), we obtain

$$W(\nu, \alpha|\ \beta, \mu\ |\lambda/2) = \delta(\nu,\ -\alpha)\ \delta(\beta,\ -\mu) \tag{2.4'}$$

3. FINITE-SIZE PROPERTIES

In this section we consider an inhomogeneous system which is related to the six-vertex model on a rotated lattice. The inhomogeneous system leads to a one-parameter family of commuting transfer matrices. A product of commuting transfer matrices can be interpreted as a transfer matrix acting on zigzag walls in the rotated system. We use an equation for commuting transfer matrices to determine their eigenvalues. We discuss finite-size properties of the rotated model from the viewpoint of the conformal field theory.

3.1. Auxiliary Vertices Method

We explain the method used to solve the six-vertex model on a rotated lattice. We start by defining an inhomogeneous system.[13] Suppose a square lattice of $M+N$ columns and $M'+N'$ rows in the natural orientation ($M = lm$, $N = ln$, $M' = l'm$, $N' = l'n$, with $M+N$ and $M'+N'$ even). We impose on it periodic boundary conditions in both directions. We also assumed that the spectral parameter v can vary from site to site. We denote the value of v for the site (j, k) by v_{jk}. Set the v_{jk} to be

$$v_{jk} = \begin{cases} -\lambda/2 & \text{for } 0 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n-1 \pmod{m+n} \\ u_0 & \text{for } m \leqslant j \leqslant m+n-1,\ 0 \leqslant k \leqslant n-1 \pmod{m+n} \\ -u_0 & \text{for } 0 \leqslant j \leqslant m-1,\ n \leqslant k \leqslant m+n-1 \pmod{m+n} \\ \lambda/2 & \text{for } m \leqslant j \leqslant m+n-1,\ n \leqslant k \leqslant m+n-1 \pmod{m+n} \end{cases} \tag{3.1}$$

with

$$u_0 = -i\bar{u}_0, \qquad -\bar{\lambda}/2 < \bar{u}_0 < \bar{\lambda}/2 \tag{3.2}$$

(Fig. 3a). The vertices $v_{jk} = -\lambda/2$ and $\lambda/2$ are auxiliary ones.

The inhomogeneous system (3.1) is related to the six-vertex model on a rotated lattice. To see this, we decompose auxiliary vertices as follows. Noting the standard initial condition (2.4), we separate the east and south edges from the west and north ones at each auxiliary vertex $v_{jk} = -\lambda/2$

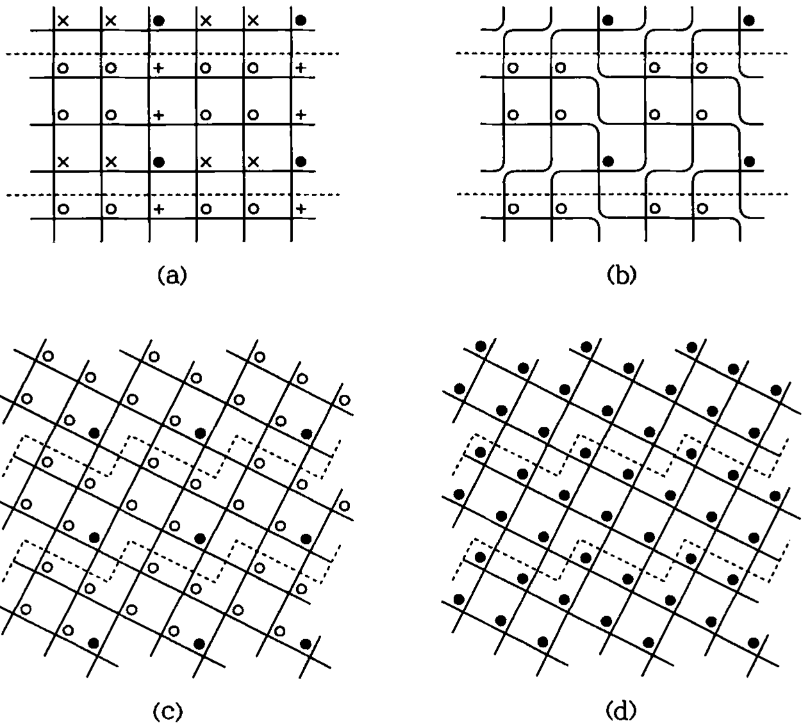

Fig. 3. (a) Inhomogeneous system (3.1) with $m = 2$ and $n = 1$. Vertices $v = -\lambda/2$ (respectively u_0, $-u_0$, $\lambda/2$) are shown by × (respectively ●, ○, +). (b) Decomposing auxiliary vertices $v = -\lambda/2$ and $\lambda/2$, we deform the lattice into a rotated one. (c) The rotated lattice is composed of two kinds of vertices $v = u_0$ and $-u_0$. (d) Using the crossing symmetry, we find the rotated model. In the rotated model $\mathbf{V}(u_0)$ is a transfer matrix between two zigzag walls, represented by broken line.

(Fig. 4a). The separation yields two types of corners. Arrows are placed so that at each corner there is one pointing in and one pointing out. We can regard a pair of edges meeting at a corner as a bonding on which an arrow is placed. Auxiliary vertices $v_{jk} = \lambda/2$ are decomposed similarly: use (2.4′) instead of (2.4), and separate the east and north edges from the west and south ones (Fig. 4b). After auxiliary vertices are decomposed (Fig. 3b), we can continuously deform the lattice into a rotated one whose rotation angle φ is given by

$$\tan \varphi = m/n \tag{3.3}$$

The rotated system consists of vertices $v_{jk} = u_0$ and $-u_0$ (Fig. 3c). The orientation of vertices $v_{jk} = u_0$ is different from that of $v_{jk} = -u_0$ in $\pi/2$

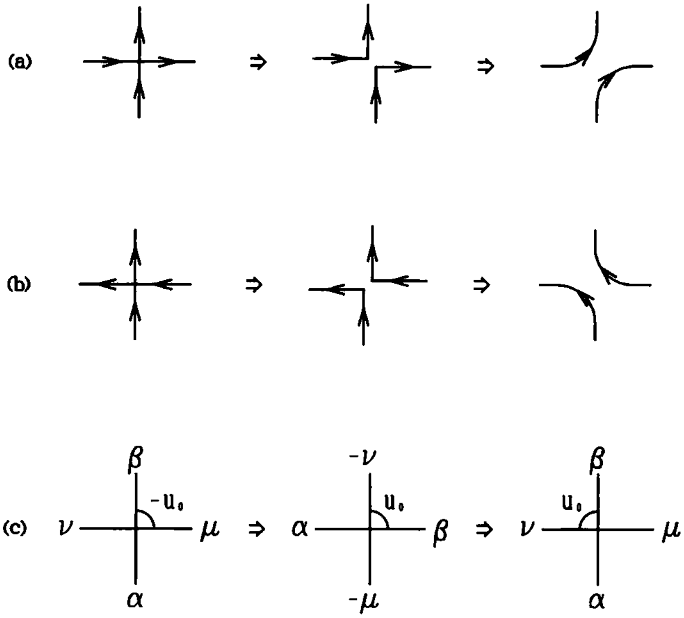

Fig. 4. Decomposition of auxiliary vertices (a) $v = -\lambda/2$ and (b) $v = \lambda/2$. (c) The orientation of vertices $v = u_0$ is rotated through $\pi/2$ by the use of the crossing symmetry.

rotation. The crossing symmetry (2.5) can be used to rotate the orientation of vertices $v_{jk} = -u_0$ through $\pi/2$, with their spectral parameter u_0 replaced by u_0 (Fig. 4c). Thus, it follows that the inhomogeneous system (3.1) is equivalent to the six-vertex model with $v = u_0$ rotated through φ with respect to the coordinate axes (Fig. 3d). We can analyze the six-vertex model rotated through φ by considering the inhomogeneous system (3.1).

The inhomogeneous system (3.1) is investigated by a commuting transfer matrices argument.[4] Let $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_{M+N}\}$ and $\beta = \{\beta_1, \beta_2, ..., \beta_{M+N}\}$ be the arrow spins on two successive rows of vertical

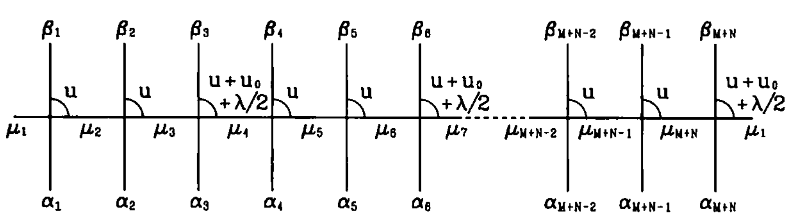

Fig. 5. One-parameter family of commuting transfer matrices with $m = 2$ and $n = 1$.

edges, and $\mu = \{\mu_1, \mu_2, ..., \mu_{M+N}\}$ the arrow spins on a row intervening between α and β (Fig. 5). A one-parameter family of transfer matrices is defined by

$$[\mathbf{T}(u)]_{\alpha,\beta} = \sum_{\mu} \prod_{i=0}^{l-1} \left[\prod_{j=i(m+n)}^{i(m+n)+m-1} W(\mu_{j+1}, \alpha_{j+1} | \beta_{j+1}, \mu_{j+2} | u) \right. \\ \left. \times \prod_{k=i(m+n)+m}^{(i+1)(m+n)-1} W(\mu_{k+1}, \alpha_{k+1} | \beta_{k+1}, \mu_{k+2} | u + u_0 + \lambda/2) \right] \tag{3.4}$$

where $\mu_{M+N+1} = \mu_1$. The Yang–Baxter relation (2.3) shows that, for all complex numbers u and u', $\mathbf{T}(u)$ and $\mathbf{T}(u')$ commute with each other, being simultaneously diagonalized. We denote the eigenvalues of $\mathbf{T}(u)$ by $\mathrm{T}(u)$.

We repeat the same argument as in Chapter 9 of ref. 4 to find a matrix $\mathbf{Q}(u)$ which satisfies the matrix equation

$$\mathbf{T}(u)\, \mathbf{Q}(u) = \Phi(u - \lambda/2)\, \mathbf{Q}(u + \lambda - \pi i) + \Phi(u + \lambda/2)\, \mathbf{Q}(u - \lambda + \pi i) \tag{3.5}$$

with

$$\Phi(u) = [\sinh(u)/\sinh(\lambda)]^M [\sinh(u + u_0 + \lambda/2)/\sinh(\lambda)]^N \tag{3.6}$$

Since $\mathbf{Q}(u)$ commutes with $\mathbf{Q}(u')$ and $\mathbf{T}(u'')$ for all complex numbers u, u', and u'', we get a functional equation for $T(u)$:

$$T(u)\, Q(u) = \Phi(u - \lambda/2)\, Q(u + \lambda - \pi i) + \Phi(u + \lambda/2)\, Q(u - \lambda + \pi i) \tag{3.7}$$

where $Q(u)$ is the eigenvalue of $\mathbf{Q}(u)$ corresponding to $T(u)$. Detailed analyses show that $Q(u)$ must be of the form

$$Q(u) = \prod_{j=1}^{\tau} \sinh(u - u_j) \tag{3.8}$$

with an integer $\tau [0 \leqslant \tau \leqslant (M+N)/2]$. The zeros u_j are determined by the condition that the rhs of (3.7) vanishes:

$$\left\{\frac{\sinh(u_j - \lambda/2)}{\sinh(u_j + \lambda/2)}\right\}^M \left\{\frac{\sinh(u_j + u_0)}{\sinh(u_j + u_0 + \lambda)}\right\}^N \\ = -\prod_{k=1}^{\tau} \frac{\sinh(u_j - u_k - \lambda)}{\sinh(u_j - u_k + \lambda)}, \qquad j = 1, 2, ..., \tau \tag{3.9}$$

The eigenvalues $T(u)$ can be calculated by solving (3.9), and then by using (3.8) in (3.7) with solutions u_j. There are many eigenvalues corresponding to different solutions of (3.9).

After the eigenvalues $T(u)$ are determined, we can get necessary information to investigate the inhomogeneous system (3.1) and hence the rotated system by letting $u = -\lambda/2$ and $-u_0$. For example, the partition function of the rotated system is calculated as

$$Z = \mathrm{Tr}[\mathbf{V}(u_0)^{l'}] = \sum_j [V_j(u_0)^{l'}]$$

$$= V_1(u_0)^{l'} \left[1 + \frac{V_2(u_0)^{l'}}{V_1(u_0)^{l'}} + \frac{V_3(u_0)^{l'}}{V_1(u_0)^{l'}} + \cdots \right] \tag{3.10a}$$

$$\mathbf{V}(u_0) = \mathbf{T}(-\lambda/2)^{n} \, \mathbf{T}(-u_0)^{m} \tag{3.10b}$$

$$V_j(u_0) = T_j(-\lambda/2)^{n} \, T_j(-u_0)^{m}, \qquad j = 1, 2, 3,... \tag{3.10c}$$

where $T_j(u)$ [or $V_j(u)$] is the jth eigenvalue of $\mathbf{T}(u)$ [or $\mathbf{V}(u)$] in decreasing order of magnitude. In the rotated system, $\mathbf{V}(u_0)$ corresponds to a transfer matrix acting on zigzag walls (Fig. 3d).

3.2. Nonlinear Integral Equations

Following the program given in Section 3.1, we calculate the asymptotic behavior of $T(u)$ as l becomes large with m and n fixed to be constants. These kinds of calculations are usually achieved by introducing a distribution function of the zeros u_j.[3, 4, 9, 12, 16–19] This approach is a very cumbersome one, however. We use another approach which was recently developed by Klümper, Batchelor, and Pearce[11] (KBP) and Klümper, Wehner, and Zittartz[20] (KWZ); see also refs. 21 and 22. Calculations are somewhat indirect. Instead of the distribution of u_j we investigate the analytic properties of the functions in (3.7) and rewrite (3.7) as nonlinear integral equations. Then the asymptotic behavior is obtained by the use of special values of Rogers dilogarithms.

As a beginning, we consider the largest eigenvalue in $\bar{u}_0 < \mathrm{Im}(u) < \bar{\lambda}/2$, which is denoted by $T_1(u)$. For convenience, we define a function $P(u)$ by

$$P(u) = \Phi(u + \lambda/2)\, Q(u - \lambda)/\Phi(u - \lambda/2)\, Q(u + \lambda) \tag{3.11}$$

and a function $h(u)$ by

$$h(u) = [1 + P(u)]/Q(u) \tag{3.12}$$

It is helpful to look at some homogeneous limits, where we can use results in refs. 3, 4, and 9–11: Considering the case $u_0 = -\lambda/2$, we determine τ as

$$\tau = (M + N)/2 \tag{3.13}$$

When l becomes large with $m=1$ and $n=0$, $|P(u)|=1$ for u on the real axis, and the zeros u_j are densely distributed on it; $T_1(u)$ is analytic and nonzero (ANZ) in $-\bar{\lambda}/2<\mathrm{Im}(u)<\bar{\lambda}/2$. In the $m=0$ and $n=1$ limit, the contour on which $|P(u)|=1$ moves into the line $\mathrm{Im}(u)=\bar{u}_0+\bar{\lambda}/2$, and the region where $T_1(u)$ is ANZ moves into $\bar{u}_0<\mathrm{Im}(u)<\bar{u}_0+\bar{\lambda}$. For general inhomogeneous cases, it is reasonable to assume the following:

(i) When l becomes large with m and n fixed to be constants, a contour C defined by $|P(u)|=1$ is in the region $0<\mathrm{Im}(u)<\bar{u}_0+\bar{\lambda}/2$, and the zeros u_j lie on the contour C.

(ii) $T_1(u)$ is ANZ in $\bar{u}_0<\mathrm{Im}(u)<\bar{\lambda}/2$.

(See, for example, ref. 13 and 23.)

The argument in KBP is repeated for $T_1(u)$. We define four functions $\alpha(x)$, $\beta(x)$, $\bar{\alpha}(x)$, and $\bar{\beta}(x)$ by

$$\alpha(x)=\beta(x)-1=1/P(x+i\varepsilon+\lambda/2) \tag{3.14a}$$

$$\bar{\alpha}(x)=\bar{\beta}(x)-1=P(x-i\delta-\lambda/2) \tag{3.14b}$$

with a real variable x and positive constants ε, δ. Note that

$$\alpha(\pm\infty)=\bar{\alpha}(\pm\infty)=1 \tag{3.15}$$

When $0<\bar{\lambda}<2\pi/3$ and $-\bar{\lambda}/2<\bar{u}_0<0$, setting ε to be $\varepsilon\sim\bar{u}_0+\bar{\lambda}/2$, and δ suitably, we consider the Fourier transform of $\alpha(x)$. The assumption (i) shows that $\Phi(u)$ is ANZ in $\bar{u}_0+\bar{\lambda}/2<\mathrm{Im}(u)<\pi$, and $Q(u)$ in $\bar{u}_0+\bar{\lambda}/2-\pi<\mathrm{Im}(u)<0$. We represent $\alpha(x)$ as

$$\alpha(x)=(-)^{\tau}\frac{\Phi(x+i\varepsilon)}{\Phi(x+i\varepsilon+i\pi+\lambda)}\frac{Q(x+i\varepsilon+3\lambda/2)}{Q(x+i\varepsilon-\lambda/2-i\pi)} \tag{3.16}$$

Taking the logarithm and second derivative of (3.16), and then the Fourier transform, we get

$$\begin{aligned}
&e^{\varepsilon k}F_k[\ln\alpha(x)]''\\
&\quad=[M+Ne^{(\bar{u}_0+\bar{\lambda}/2)k}-Me^{(\bar{\lambda}-\pi)k}-Ne^{(\bar{u}_0+3\bar{\lambda}/2-\pi)k}]\frac{k}{1-e^{-k\pi}}\\
&\qquad+[e^{3\bar{\lambda}k/2}-e^{(\pi-\bar{\lambda}/2)k}]F_k[\ln Q(x)]''
\end{aligned} \tag{3.17}$$

where we denote by $F_k[f(x)]$ the Fourier transform of a function $f(x)$:

$$F_k[f(x)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)\,e^{-kx}\,dx \tag{3.18}$$

From the assumptions (i) and (ii) it follows that $h(u)$ is ANZ in $\bar{u}_0 < \mathrm{Im}(u) < \bar{\lambda}/2$. On a path with the imaginary part $\varepsilon - \bar{\lambda}/2, h(u)$ is represented as

$$h(x + i\varepsilon - i\bar{\lambda}/2) = \beta(x)/Q(x + i\varepsilon - i\bar{\lambda}/2)\, \alpha(x) \tag{3.19}$$

Take the Fourier transform of the second logarithmic derivative of (3.19). It follows that

$$\begin{aligned} &e^{\bar{\lambda}k/2} F_k[\ln h(x)]'' \\ &\quad = -e^{\bar{\lambda}k/2} F_k[\ln Q(x)]'' + e^{\varepsilon k} F_k[\ln \beta(x)]'' - e^{\varepsilon k} F_k[\ln \alpha(x)]'' \end{aligned} \tag{3.20}$$

On a path with the imaginary part $-\delta + \bar{\lambda}/2, h(u)$ is given by

$$h(x - i\delta + i\bar{\lambda}/2) = (-)^{\tau} \bar{\beta}(x)/Q(x - i\delta + i\bar{\lambda}/2 - i\pi) \tag{3.21}$$

From (3.21) we obtain another formula for $F_k[\ln h(x)]''$:

$$e^{-\bar{\lambda}k/2} F_k[\ln h(x)]'' = e^{-\delta k} F_k[\ln \bar{\beta}(x)]'' - e^{(\pi - \bar{\lambda}/2)k} F_k[\ln Q(x)]'' \tag{3.22}$$

We solve (3.17), (3.20), and (3.22) for $F_k[\ln \alpha(x)]''$ in terms of $F_k[\ln \beta(x)]''$ and $F_k[\ln \bar{\beta}(x)]''$. It is found that

$$\begin{aligned} F_k[\ln \alpha(x)]'' &= [M + N e^{(\bar{u}_0 + \bar{\lambda}/2)k}] \frac{k e^{-\varepsilon k}}{1 + e^{-\bar{\lambda}k}} \\ &\quad + \frac{\sinh[(\pi/2 - \bar{\lambda})k]}{2\cosh(\bar{\lambda}k/2)\sinh[(\pi - \bar{\lambda})\,k/2]} \\ &\quad \times \{F_k[\ln \beta(x)]'' - e^{(\bar{\lambda} - \delta - \varepsilon)k} F_k[\ln \bar{\beta}(x)]''\} \end{aligned} \tag{3.23}$$

Applying the inverse Fourier transform to (3.23), we get

$$\begin{aligned} [\ln \alpha(x)]'' &= M \left\{ \ln \left[\tanh \frac{\pi(x + i\varepsilon)}{2\bar{\lambda}} \right] \right\}'' \\ &\quad + N \left\{ \ln \left[\tanh \frac{\pi(x + i\varepsilon + u_0 + \lambda/2)}{2\bar{\lambda}} \right] \right\}'' \\ &\quad + \int_{-\infty}^{\infty} \{F(x - y)[\ln \beta(y)]'' - F(x + i\varepsilon - y - i\bar{\lambda} + i\delta) \\ &\quad \times [\ln \bar{\beta}(y)]''\}\, dy \end{aligned} \tag{3.24}$$

with

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh[(\pi - 2\bar{\lambda})\,k/2]}{2\cosh(\bar{\lambda}k/2)\sinh[(\pi - \bar{\lambda})\,k/2]} e^{ikx}\, dk \tag{3.25}$$

Integrating (3.24); twice gives

$$\ln \alpha(x) = M \ln\left[\tanh \frac{\pi(x+i\varepsilon)}{2\bar{\lambda}}\right] + N \ln\left[\tanh \frac{\pi(x+i\varepsilon+u_0+\lambda/2)}{2\bar{\lambda}}\right]$$
$$+\int_{-\infty}^{\infty} [F(x-y)\ln\beta(y) - F(x+i\varepsilon-y-i\bar{\lambda}+i\delta)\ln\bar{\beta}(y)]\,dy$$
$$+Cx+D \tag{3.26}$$

The asymptotics (3.15) shows that

$$C=D=0 \tag{3.27}$$

In a similar way we solve (3.17),(3.20), and (3.22) to find an integral equation for $\ln Q(x)$. We rewrite (3.7) in the form

$$T_1(x+i\varepsilon-i\bar{\lambda}/2) = \Phi(x+i\varepsilon-i\bar{\lambda})\,\frac{Q(x+i\varepsilon+i\bar{\lambda}/2-i\pi)}{Q(x+i\varepsilon-i\bar{\lambda}/2)}\,\beta(x) \tag{3.28}$$

Taking the logarithm of (3.28), and using the integral equation for $\ln Q(x)$ in it, we obtain

$$\ln T_1(x+i\varepsilon-i\bar{\lambda}/2)$$
$$= \ln \Phi(x+i\varepsilon-i\bar{\lambda}) - Mi\int_{-\infty}^{\infty} \frac{\sinh[(\pi-\bar{\lambda})\,k/2]\,\sin[(x+i\varepsilon)k]}{2k\sinh(\pi k/2)\,\cosh(\bar{\lambda}k/2)}\,dk$$
$$-Ni\int_{-\infty}^{\infty} \frac{\sinh[(\pi-\bar{\lambda})\,k/2]\,\sin[(x+i\varepsilon+u_0+\lambda/2)k]}{2k\sinh(\pi k/2)\cosh(\bar{\lambda}k/2)}\,dk$$
$$+\frac{i}{2\bar{\lambda}}\int_{-\infty}^{\infty} \frac{\ln\beta(y)}{\sinh[\pi(x-y+i0)/\bar{\lambda}]}\,dy$$
$$+\frac{i}{2\bar{\lambda}}\int_{-\infty}^{\infty} \frac{\ln\bar{\beta}(y)}{\sinh[\pi(x-y+i\varepsilon+i\delta)/\bar{\lambda}]}\,dy \tag{3.29}$$

The integral equations (3.26) and (3.29) are exact for sufficiently large (but finite) l.

As l becomes large, we examine the consistency of the analysis by the use of leading terms in (3.26) and (3.29). In the $l\to\infty$ limit, (3.29) shows that

$$\ln T_1(u) \sim M\ln\kappa(u) + N\ln\kappa(u+u_0+\lambda/2) \tag{3.30}$$

for $\bar{u}_0 < \mathrm{Im}(u) < \bar{\lambda}/2$ with

$$\ln \kappa(u) = \ln \frac{\sinh(u+\lambda/2)}{\sinh \lambda} - i \int_{-\infty}^{\infty} \frac{\sinh[(\pi-\bar{\lambda})\,k/2]\,\sin[(u-\lambda/2)k]}{2k\,\sinh(\pi k/2)\,\cosh(\bar{\lambda}k/2)}\,dk$$
$$= \ln \frac{\sinh(-u+\lambda/2)}{\sinh \lambda} - i \int_{-\infty}^{\infty} \frac{\sinh[(\pi-\bar{\lambda})\,k/2]\,\sin[(-u-\lambda/2)k]}{2k\,\sinh(\pi k/2)\,\cosh(\bar{\lambda}k/2)}\,dk \qquad (3.31)$$

The asymptotic form (3.30) satisfies the analytic property in the assumption (ii). When l becomes large, the leading behavior is given by the first and second terms on the rhs of (3.26). We estimate $P(u)$ as

$$P(u) \sim \left[\tanh \frac{\pi(u+\lambda/2)}{2\bar{\lambda}}\right]^M \left[\tanh \frac{\pi(u+u_0+\lambda)}{2\bar{\lambda}}\right]^N \qquad (3.32)$$

Equation (3.32) is consistent with the assumption (i). If we denote by $\rho(u)$ the density function of u_j along the contour C, we can calculate $\rho(u)$ from (3.32) as

$$\rho(u) = \frac{1}{2\bar{\lambda}}\left[\frac{M}{\cosh(\pi u/\bar{\lambda})} + \frac{N}{\cosh[\pi(u+u_0+\lambda/2)/\bar{\lambda}]}\right] \qquad (3.33)$$

Using (3.33) in (3.7), we can rederive (3.30). These facts justify the argument from the assumptions (i) and (ii) to (3.29).

To find finite-size corrections of $T_1(u)$, we introduce following functions:

$$a_\pm(x) = A_\pm(x) - 1 = \lim_{l\to\infty} \alpha[\pm\bar{\lambda}(x+\ln l)/\pi] \qquad (3.34a)$$

$$\bar{a}_\pm(x) = \bar{A}_\pm(x) - 1 = \lim_{l\to\infty} \bar{\alpha}[\pm\bar{\lambda}(x+\ln l)/\pi] \qquad (3.34b)$$

Equation (3.26) with (3.27) shows that

$$\ln a_\pm(x) = -2[m + n e^{\mp\pi(u_0+\lambda/2)/\bar{\lambda}}]\, e^{-x \mp i\varepsilon\pi/\bar{\lambda}} + F_1 * \ln A_\pm(x) - F_2 * \ln \bar{A}_\pm(x) \qquad (3.35a)$$

$$\ln \bar{a}_\pm(x) = -2[m + n e^{\mp\pi(u_0+\lambda/2)/\bar{\lambda}}]\, e^{-x \mp i\delta\pi/\bar{\lambda}} + F_2^* * \ln A_\pm(x) - F_1 * \ln \bar{A}_\pm(x) \qquad (3.35b)$$

where

$$F_1(x) = (\bar{\lambda}/\pi)\, F(\bar{\lambda}x/\pi)$$
$$F_2(x) = (\bar{\lambda}/\pi)\, F[\bar{\lambda}x/\pi \pm i(\varepsilon + \delta - \bar{\lambda})] \tag{3.36}$$

and $f * g(x)$ is the convolution of two functions $f(x)$ and $g(x)$:

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy \tag{3.37}$$

From (3.15) and (3.35), we deduce that

$$a_{\pm}(\infty) = 1, \qquad a_{\pm}(-\infty) = 0 \tag{3.38a}$$

$$\bar{a}_{\pm}(\infty) = 1, \qquad \bar{a}_{\pm}(-\infty) = 0 \tag{3.38b}$$

It follows from (3.29) that

$$\begin{aligned}
\ln T_1(x + i\varepsilon - i\bar{\lambda}/2)
&= M \ln \kappa(x + i\varepsilon - i\bar{\lambda}/2) + N \ln \kappa(x + i\varepsilon + u_0 - i\bar{\lambda}) \\
&\quad - \frac{i}{\pi l} e^{\pi(x+i\varepsilon)/\bar{\lambda}} \int_{-\infty}^{\infty} [\ln A_+(y)\, e^{-y - i\varepsilon\pi/\bar{\lambda}} + \ln \bar{A}_+(y)\, e^{-y + i\delta\pi/\bar{\lambda}}]\, dy \\
&\quad + \frac{i}{\pi l} e^{-\pi(x+i\varepsilon)/\bar{\lambda}} \int_{-\infty}^{\infty} [\ln A_-(y)\, e^{-y + i\varepsilon\pi/\bar{\lambda}} + \ln \bar{A}_-(y)\, e^{-y - i\delta\pi/\bar{\lambda}}]\, dy \\
&\quad + o(1/l)
\end{aligned} \tag{3.39}$$

The integrals on the rhs of (3.39) are expressed in terms of Rogers dilogarithms. From (3.35) we obtain

$$\begin{aligned}
&2[m + n e^{\mp\pi(u_0 + \lambda/2)/\bar{\lambda}}] \int_{-\infty}^{\infty} e^{-x \mp i\varepsilon\pi/\bar{\lambda}} \{\ln A_{\pm}(x) + [\ln A_{\pm}(x)]'\}\, dx \\
&\quad + 2[m + n e^{\mp\pi(u_0 + \lambda/2)/\bar{\lambda}}] \int_{-\infty}^{\infty} e^{-x \pm i\delta\pi/\bar{\lambda}} \{\ln \bar{A}_{\pm}(x) + [\ln \bar{A}_{\pm}(x)]'\}\, dx \\
&= \int_{-\infty}^{\infty} \{[\ln a_{\pm}(x)]' \ln A_{\pm}(x) - \ln a_{\pm}(x) [\ln A_{\pm}(x)]'\}\, dx \\
&\quad + \int_{-\infty}^{\infty} \{[\ln \bar{a}_{\pm}(x)]' \ln \bar{A}_{\pm}(x) - \ln \bar{a}_{\pm}(x) [\ln \bar{A}_{\pm}(x)]'\}\, dx
\end{aligned} \tag{3.40}$$

Integrating the lhs by parts and using (3.34), we get

$$\begin{aligned} &4[m+ne^{\mp\pi(u_0+\lambda/2)/\bar{\lambda}}] \\ &\quad\times\int_{-\infty}^{\infty} dx\,[e^{-x\mp i\varepsilon\pi/\bar{\lambda}}\ln A_{\pm}(x)+e^{-x\pm i\delta\pi/\bar{\lambda}}\ln \bar{A}_{\pm}(x)] \\ &=\int_{a_{\pm}(-\infty)}^{a_{\pm}(\infty)} da_{\pm}\,[a_{\pm}^{-1}\ln(1+a_{\pm})-(1+a_{\pm})^{-1}\ln a_{\pm}] \\ &\quad+\int_{\bar{a}_{\pm}(-\infty)}^{\bar{a}_{\pm}(\infty)} d\bar{a}_{\pm}[\bar{a}_{\pm}^{-1}\ln(1+\bar{a}_{\pm})-(1+\bar{a}_{\pm})^{-1}\ln\bar{a}_{\pm}] \end{aligned} \tag{3.41}$$

Substitute the asymptotics (3.38) into (3.41). It follows that

$$\begin{aligned} &4[m+ne^{\mp\pi(u_0+\lambda/2)/\bar{\lambda}}] \\ &\quad\times\int_{-\infty}^{\infty} dx\,[e^{-x\mp i\varepsilon\pi/\bar{\lambda}}\ln A_{\pm}(x)+e^{-x\pm i\delta\pi/\bar{\lambda}}\ln \bar{A}_{\pm}(x)] \\ &=4L_{+}(1)=4L(1/2)=\pi^2/3 \end{aligned} \tag{3.42}$$

where $L_{+}(a)$ is a dilogarithmic function defined by

$$L_{+}(a)=(1/2)\int_0^a db\,[b^{-1}\ln(1+b)-(1+b)^{-1}\ln b] \tag{3.43}$$

and $L(a)$ is the Rogers dilogarithm:[24]

$$L(a)=-(1/2)\int_0^a db\,[b^{-1}\ln(1-b)+(1-b)^{-1}\ln b] \tag{3.44}$$

The dilogarithmic functions are related to each other by

$$L_{+}(a)=L[a/(1+a)] \tag{3.45}$$

Using (3.42) in (3.39), and after some calculations, we find that

$$\begin{aligned} \ln T_1(u)&=M\ln\kappa(u)+N\kappa(u+u_0+\lambda/2) \\ &\quad+\frac{\pi}{12l}\frac{e^{\pi u/\bar{\lambda}}}{m+ne^{-\pi(u_0+\lambda/2)/\bar{\lambda}}}+\frac{\pi}{12l}\frac{e^{-\pi u/\bar{\lambda}}}{m+ne^{\pi(u_0+\lambda/2)/\bar{\lambda}}}+o(1/l) \end{aligned} \tag{3.46}$$

for $\bar{u}_0<\mathrm{Im}(u)<\bar{\lambda}/2$. In the first stage of the calculation we assume that $0<\bar{\lambda}<2\pi/3$ and that $-\bar{\lambda}/2<\bar{u}_0<0$. For regimes $0<\bar{\lambda}<2\pi/3$, $0<\bar{u}_0<\bar{\lambda}/2$ and $2\pi/3<\bar{\lambda}<\pi$, $-\bar{\lambda}/2<\bar{u}_0<\bar{\lambda}/2$, the same method is applied with some deformations of paths of Fourier integrals. We find that the result (3.46) is valid throughout the entire gapless regime $0<\bar{\lambda}<\pi$, $-\bar{\lambda}/2<\bar{u}_0<\bar{\lambda}/2$.

The next largest eigenvalues in $\bar{u}_0 < \mathrm{Im}(u) < \bar{\lambda}/2$ are calculated similarly. We repeat almost the same argument as KWZ. When l becomes large with m and n fixed to be constants, we obtain

$$\ln T_{x,s}(u) = M \ln \kappa(u) + N\kappa(u + u_0 + \lambda/2) + p_0 \pi i$$
$$+ \frac{\pi}{l}\left(-x + \frac{1}{12} - s\right) \frac{e^{\pi u/\bar{\lambda}}}{m + n e^{-\pi(u_0 + \lambda/2)/\bar{\lambda}}}$$
$$+ \frac{\pi}{l}\left(-x + \frac{1}{12} + s\right) \frac{e^{-\pi u/\bar{\lambda}}}{m + n e^{\pi(u_0 + \lambda/2)/\bar{\lambda}}} + o\left(\frac{1}{l}\right) \qquad (3.47)$$

where p_0 corresponds to the lattice momentum, whose explicit form is not important here; the scaling dimension x and the spin s are given by

$$x = \frac{1 - \bar{\lambda}/\pi}{2} S^2 + \frac{\bar{S}^2}{2(1 - \bar{\lambda}/\pi)} + k + \bar{k} \qquad (3.48a)$$

$$s = S\bar{S} + k - \bar{k} \qquad (3.48b)$$

with an integer $\bar{S}$ and nonnegative integers $S, k, \bar{k}$ (see also ref. 16 and 19); the number of zeros in (3.8) is related to S by

$$\tau = (M + N)/2 - S \qquad (3.49)$$

3.3. Spatial Anisotropy and Conformal Invariance

Now, using the results calculated in Section 3.2, we investigate the six-vertex model on a rotated lattice. Emphasis is placed on finite-size corrections of the partition function Z. Finite-size properties of the model are discussed from the viewpoint of the conformal field theory.

Set $u = -\lambda/2$ and $-u_0$ in (3.46) and (3.47). Substituting $T(-\lambda/2)$ and $T(-u_0)$ into (3.10c), we find that

$$\ln V_1^{l'}(u_0) \sim (MM' + NN') \ln \kappa(u_0) + (\pi/6)(l'/l)/\Gamma^2 \qquad (3.50a)$$

$$\ln \left[\frac{V_{x,s}(u_0)}{V_1(u_0)}\right]^{l'} \sim -2\pi x \frac{l'}{l} \frac{1}{\Gamma^2} - 2\pi\, is \frac{l'}{l} \frac{\Delta}{\Gamma^2} \qquad (\mathrm{mod}\ 2\pi i) \qquad (3.50b)$$

where

$$\Gamma^2 = \gamma^2 \cos^2\theta + \gamma^{-2} \sin^2\theta \qquad (3.51a)$$

$$\Delta = (\gamma^2 - \gamma^{-2}) \sin\theta \cos\theta \qquad (3.51b)$$

with

$$\gamma^2 = \tan\left(\frac{\pi \bar{u}_0}{2\bar{\lambda}} + \frac{\pi}{4}\right), \qquad \theta = \varphi + \frac{\pi}{4} \tag{3.52}$$

(see, for example, ref. 25).

The bulk contribution is given by the first term on the rhs of (3.50a). The per-site free energy f is calculated as

$$f = -\ln \kappa(u_0) \tag{3.53}$$

It is shown that f is independent of the rotation angle φ, which is an extension of the numerical result by LP.

We define the leading finite-size correction of the partition function Z by

$$Z_c = \lim_{l, l' \to \infty} Z/\kappa(u_0)^{ll'} \tag{3.54}$$

where the limit is taken with the ratio l'/l fixed.[7, 13, 26] If the ratio l'/l is small, the partition function Z can be expanded as (3.10a).

We consider the case of isotropic interactions, where $u_0 = 0$. From (3.52) it follows that $\gamma^2 = 1$. Substituting $\gamma^2 = 1$ into (3.51) gives $\Gamma^2 = 1$ and $\Delta = 0$. Within the validity of the expansion, it is found that Z_c is a function of the ratio l'/l but is independent of the rotation angle φ. This result shows that the system is invariant under dilatations and rotations. Note that a conformal transformation can be regarded as a combination of translations, dilatations, and rotations locally. Though the global transformations cannot be generalized to local ones directly, it is suggested that the system is conformably invariant.

The conformal field theory shows that Z_c must be written in the form[7]

$$Z_c = \mathrm{Tr}(q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}) \tag{3.55}$$

where

$$q = \exp(2\pi i \tau), \qquad \bar{q} = \exp(-2\pi i \bar{\tau}) \tag{3.56}$$

with

$$\tau = il'/l, \qquad \bar{\tau} = -il'/l \tag{3.57}$$

and L_0, $\bar{L}_0$ are Virasoro generators; $L_0 + \bar{L}_0$ generates translations along the vertical direction, and $i(L_0 - \bar{L}_0)$ along the horizontal direction. The expansion (3.10a) with (3.50) is consistent with (3.55). It follows from (3.50a) that the central charge $c = 1$. The scaling dimensions of low-lying excitations are given by (3.50b) with (3.48a).

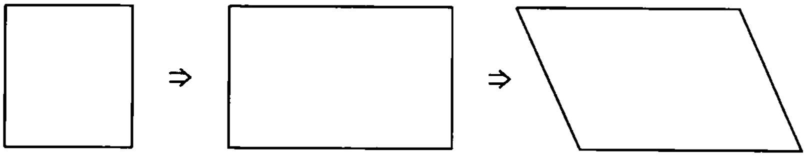

Fig. 6. For most anisotropic systems conformal invariance is restored by deforming the geometry of the lattice.

For most anisotropic systems τ and $\bar{\tau}$ are replaced by

$$\tau = (l'/l)(i + \Delta)/\Gamma^2, \qquad \bar{\tau} = (l'/l)(-i + \Delta)/\Gamma^2 \tag{3.58}$$

Equation (3.58) shows that the anisotropic system has the same finite-size correction Z_c as the conformal invariant one if the geometry of the anisotropic system is deformed as follows (Fig. 6):[13] stretch the original rectangular lattice along either of the two coordinate axes ($\tau \to \tau\Gamma^2$, $\bar{\tau} \to \bar{\tau}\Gamma^2$), shear it into a parallelogram ($\tau \to \tau - \Delta l'/l$, $\bar{\tau} \to \bar{\tau} - \Delta l'/l$), then impose on it periodic boundary conditions in both directions. Locally, the deformation corresponds to an anisotropic resealing of length by an amount of γ^2 along a direction rotated through θ from the coordinate axes. Note that γ^2 is independent of the rotation angle θ. We find that conformal invariance of the anisotropic system is restored by the anisotropic resealing.

4. SUMMARY AND DISCUSSION

We analyzed the six-vertex model on a square lattice rotated through an arbitrary angle φ with respect to the coordinate axes by a new method. We introduced auxiliary vertices into a rotated system to relate it to an inhomogeneous one in the natural orientation $\varphi = 0$. The inhomogeneous system led to a one-parameter family of commuting transfer matrices. It was shown that a product of commuting transfer matrices can be interpreted as a transfer matrix acting on zigzag walls in the rotated model. We solved an equation for commuting transfer matrices to determine their eigenvalues.

We considered the gapless regime of the six-vertex model. Supposing that the model is on a torus of size $l \times l'$, we investigated the leading finite-size correction Z_c of the partition function. The finite-size correction Z_c was expanded by the use of some largest eigenvalues of the zigzag-wall transfer matrix. When interactions are isotropic, it was found that Z_c is a function of the ratio l'/l, but is independent of the rotation angle φ. From this fact it was suggested that the model is conformably invariant. For most anisotropic systems, reflecting a breakdown of the rotational invariance, extra factors Γ^2 and Δ appear in Z_c. We showed that the rotational invariance is restored by an anisotropic rescaling of length by an amount of γ^2 along a direction rotated through $\theta(=\varphi+\pi/4)$ with respect to the coordinate axes. The factor γ^2 was determined as a function of the spectral parameter $\bar{u}_0$ and the crossing parameter $\bar{\lambda}$ in (3.52). The problem of the rotational invariance has been discussed in refs. 12 and 27 for two orientations $\varphi=0$ and $\pi/4$. Our results are consistent with the argument in refs. 12 and 27. We note that essentially the same expression for γ^2 has been found in analyses for hard squares, magnetic hard squares, and the q-state Potts models.[13, 27] It is expected that (3.52) is a general expression for solvable models on square lattices.

It is known that, if a square lattice is drawn diagonally, the geometry is convenient to consider relations between the six-vertex model and the XXZ-Heisenberg chain.[4, 14, 28–30] Analyses of finite-size properties in the critical XXZ chain have been undertaken by many authors; see, for example, refs. 17–19. It is useful, however, to reexamine these properties by the use of eigenvalues of the diagonal-to-diagonal transfer matrix of the six-vertex model. Here, fixing the rotation angle to be $\theta=\varphi+\pi/4=\pi/2$ with $m=n=1$, we investigate the XXZ chain. Substituting $\theta=\pi/2$ into (3.51) and (3.58) gives

$$\tau=i(l'/l)\gamma^2, \qquad \bar{\tau}=-i(l'/l)\,\gamma^2 \tag{4.1}$$

with

$$\gamma^2=\tan\left(\frac{\pi\bar{u}_0}{2\bar{\lambda}}+\frac{\pi}{4}\right) \tag{4.2}$$

We define the Hamiltonian $\mathbf{H}$ of the XXZ chain of $2l$ sites by

$$\mathbf{H}=\sum_{j=1}^{2l}\left(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\cos\bar{\lambda}\,\sigma_j^z\sigma_{j+1}^z\right) \tag{4.3}$$

where $\sigma^x, \sigma^y, \sigma^z$ are Pauli spin matrices and σ_{2l+1} is to be interpreted as σ_1. The diagonal-to-diagonal transfer matrix $\mathbf{V}$ is expanded around $\bar{u}_0=-\bar{\lambda}/2$ as

$$\begin{aligned}\mathbf{V}&=\mathbf{I}-(2\sin\bar{\lambda})^{-1}(\cos\bar{\lambda}\,\mathbf{I}+\mathbf{H})\,\delta\bar{u}_0+\cdots\\ \bar{u}_0&=-\bar{\lambda}/2+\delta\bar{u}_0\end{aligned} \tag{4.4}$$

where $\mathbf{I}$ is the identity matrix. From the partition function Z of the six-vertex model, we can determine the partition function Z_{H} of the XXZ chain by setting $\delta\bar{u}_0=2(\beta/l')\sin\bar{\lambda}$, and then by taking the $l'\to\infty$ limit.

When $\beta \gg 2l \gg 1$, we can use the expansion (3.10a) with (3.50). We calculate the leading behavior of the free energy F_H of the XXZ chain as

$$F_H = 2l\beta\varepsilon - (\beta/2l)\, v(\pi c/6) + \cdots \tag{4.5}$$

where the per-site ground-state energy ε is given by

$$\varepsilon = \cos \bar{\lambda} - \sin \bar{\lambda} \int_{-\infty}^{\infty} \frac{\sinh[(\pi - \bar{\lambda})\, k/2]}{\sinh(\pi k/2) \cosh(\bar{\lambda} k/2)}\, dk \tag{4.6}$$

The $\beta/2l$ correction term shows that the central charge is $c = 1$. The effective light velocity v is determined by the derivative of γ^2 at $\bar{u}_0 = -\bar{\lambda}/2$ as

$$v = (2\pi/\bar{\lambda}) \sin \bar{\lambda} \tag{4.7}$$

In (3.55) the finite-size correction Z_c of the six-vertex model was represented as a function of τ and $\bar{\tau}$. Note that the rescaling factor γ^2 becomes 0 in the $\bar{u}_0 \to -\bar{\lambda}/2$ limit, where interactions are extremely anisotropic. Taking the $l' \to \infty$ limit has the effect of replacing the ratio l'/l of the six-vertex model by the ratio $\beta/2l$ of the XXZ chain. We denote by $Z_{H;c}$ the leading finite-size correction of the partition function of the XXZ model. Within the validity of the expansion (3.10a), we find that

$$Z_{H;c} = \lim_{l' \to \infty} Z_c(\tau, \bar{\tau}) = Z_c(\tau_H, \bar{\tau}_H) \tag{4.8}$$

where the limit is taken with $\delta \bar{u}_0 = 2(\beta/l') \sin \bar{\lambda}$ and

$$\tau_H = \lim_{l' \to \infty} \tau = i(\beta/2l)\, v, \qquad \bar{\tau}_H = \lim_{l' \to \infty} \bar{\tau} = -i(\beta/2l)\, v \tag{4.9}$$

When $2l \gg \beta \gg 1$, it is convenient to see the lattice from a $\pi/2$ rotated frame.[31] This is achieved by the change in the spectral parameter $\bar{u}_0$ into $-\bar{u}_0$. In (4.1) γ^2 is replaced by

$$\gamma^2 = \cot\left(\frac{\pi \bar{u}_0}{2\bar{\lambda}} + \frac{\pi}{4}\right) \tag{4.10}$$

We set $\bar{u}_0 = -\bar{\lambda}/2 + \delta \bar{u}_0$ with $\delta \bar{u}_0 = 2(\beta/l) \sin \bar{\lambda}$ and take the $l \to \infty$. For $2l' \gg \beta \gg 1$, F_H is calculated as[32]

$$F_H = 2l'\beta\varepsilon - (2l'/\beta)\, v^{-1}(\pi c/6) + \cdots \tag{4.11}$$

The $2l'/\beta$ correction term in (4.11) is also found from (4.8) by assuming the invariance of Z_c under the modular transformation $\tau \to -1/\tau$, $\bar{\tau} \to -1/\bar{\tau}$.

Finally, it may be worth pointing out that the auxiliary vertices method has various applications besides analyses for rotated models. For example, we can show that the method is directly applicable to transfer matrix calculations for solvable models on triangular, honeycomb, and Kagomé lattices. These analyses will be reported in further publications.

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REFERENCES

1. A. A. Litvin and V. B. Priezzhev, *J. Stat. Phys.* **60**:307–321 (1990).
2. H. Bethe, *Z. Phys.* **71**:205–226 (1931).
3. E. H. Lieb, *Phys. Rev. Lett.* **18**:692–694 (1967); *Phys. Rev.* **162**:162–172 (1967).
4. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
5. A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *J. Stat. Phys.* **34**:763–774 (1984); *Nucl. Phys. B* **241**:333–380 (1984).
6. J. L. Cardy, In *Phase Transitions and Critical Phenomena*, Vol. 11, C. Domb and J. L. Lebowitz, eds. (Academic Press, London, 1987).
7. C. Itzykson and J. M. Drouffe, *Statistical Field Theory* (Cambridge University Press, Cambridge, 1989), Chapter 9.
8. H. W. Blöte, J. L. Cardy, and M. P. Nightingale, *Phys. Rev. Lett.* **56**:742–745 (1986).
9. H. J. de Vega and M. Karowski, *Nucl. Phys. B* **285**:619–638 (1987).
10. A. Klümper and M. T. Batchelor, *J. Phys. A: Math. Gen.* **23**:L189–L195 (1990).
11. A. Klümper, M. T. Batchelor, and P. A. Pearce, *J. Phys. A: Math. Gen.* **24**:3111–3133 (1991).
12. A. L. Owczarek and R. J. Baxter, *J. Phys. A: Math. Gen.* **22**:1141–1165 (1989).
13. M. Fujimoto, *J. Phys. A: Math. Gen.* **27**:5101–5119 (1994).
14. M. Wadati and Y. Akutsu, *Prog. Theor. Phys. Suppl.* **94**:1–41 (1988).
15. M. Wadati, T. Deguchi, and Y. Akutsu, *Phys. Rep.* **180**:247–332 (1989).
16. M. Karowski, *Nucl. Phys. B* **300[FS22]**:473–499 (1988).
17. C. J Harmer, *J. Phys. A: Math. Gen.* **18**:L1133–L1137 (1985); *J. Phys. A: Math. Gen.* **19**:3335–3351 (1986).
18. F. C. Alcaraz, M. N. Barber, and M. T. Batchelor, *Ann. Phys.* **182**:280–343 (1988).
19. F. Woynarovich, *Phys. Rev. Lett.* **59**:259–261 (1987).
20. A. Klümper, T. Wehner, and J. Zittartz, *J Phys. A: Math. Gen.* **26**:2815–2827 (1993).
21. P. A. Pearce and A. Klümper, *Phys. Rev. Lett.* **66**:974–977 (1991).
22. A. Klümper and P. A. Pearce, *J. Stat. Phys.* **64**:13–76 (1991).
23. M. Fujimoto, *J. Stat. Phys.* **67**:123–154 (1992).
24. L. Lewin, *Dilogarithms and Associated Functions* (McDonald, London, 1958).
25. M. P. Nightingale and H. W. Blöte, *J. Phys. A: Math. Gen.* **16**:L657–L664 (1983).
26. J. L. Cardy, *Nucl. Phys. B* **270**:186–204 (1986).

27. D. Kim and P. A. Pearce, *J. Phys. A: Math. Gen.* **20**:L451–L456 (1987).
28. B. Sutherland, *J. Math. Phys.* **11**:3183–3186 (1970).
29. H. N. V. Temperly and E. H. Lieb, *Proc. R. Soc. Lond. A* **322**:251–280 (1971).
30. H. E. Lieb and F. Y. Wu, In *Phase Transitions and Critical Phenomena*, Vol. 1, C. Domb and M. S. Green, eds. (Academic Press, London, 1972).
31. J. Suzuki, Y. Akutsu, and M. Wadati, *J. Phys. Soc. Jpn.* **59**:2667–2680 (1990).
32. I. Affleck, *Phys. Rev. Lett.* **56**:746–748 (1986).